Malaysian Journal of Mathematical Sciences 9(2): 317-323 (2015)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

# Endomorphism Rings of Semi-Injective Coretractable Modules

**Tayyebeh Amouzegar**

*Department of Mathematics,*
*Quchan University of Advanced Technology, Quchan, Iran*

*E-mail: t.amouzgar@qiet.ac.ir*

## ABSTRACT

Let $M$ be a semi-injective coretractable module with endomorphism ring $S$. . It is shown that if $M$ is an UCC module and $J \leq_e \ell_S(r_M(J))$ for every $J \leq S_S$, then $M$ is a lifting module if and only if $S_S$ is extending.

Keywords: coretractable module, semi-injective module, endomorphism ring.

## 1. INTRODUCTION

Extending modules and lifting modules have been intensively studied throughout the last two decades, due to their important role played in ring and module theory. Recall that an $R$-module $M$ is an *extending* module if for every submodule $A$ of $M$ there exists a direct summand $B$ of $M$ such that $A \leq_e B$ (Amini, 2009). A ring $R$ is a right extending ring if $R_R$ is an extending module. Dually, a module $M$ is called a *lifting* module if, every submodule $N$ of $M$ can be written in the form $N = A \oplus D$ where $A$ is a direct summand of $M$ and $D \ll M$ (Clark, 2006). In this note, we find conditions under which the endomorphism ring of a lifting module to be extending.

Throughout this paper, $R$ will denote an arbitrary associative ring with identity, $M$ a unitary right $R$-module and $S = End_R(M)$ the ring of all $R$-endomorphisms of $M$. We will use the notation $N \leq_e M$ to indicate that $N$ is essential in $M$ (i.e., $N \cap L \neq 0 \; \forall 0 \neq L \leq M$); $N \langle\langle M$ means that

$N$ is small in $M$ (i.e. $\forall L \lneq M, L+N \neq M$). For all $I \subseteq S$, the left and right annihilators of $I$ in $S$ are denoted by $\ell_S(I)$ and $r_S(I)$, respectively. We also denote $r_M(I)=\{x \in M \mid Ix=0\}$, for $I \subseteq S$; $\ell_S(N)=\{\phi \in S \mid \phi(N)=0\}$, for $N \subseteq M$.

An $R$-module $M$ is called *semi-injective* if for any $f \in S$, $Sf=\ell_S(\ker(f))=\ell_S(r_M(f))$ (equivalently, for any monomorphism $f: N \to M$, where $N$ is a factor module of $M$, and for any homomorphism $g: N \to M$, there exists $h: M \to M$ such that $hf=g$) (Wisbauer, 1999). Clearly every quasi-injective module is semi-injective. Let $M$ be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then $B$ is called a *cosmall* submodule of $A$ in $M$ (denoted by $B \xrightarrow{cs} A$). Recall that a submodule $A$ of $M$ is called *coclosed* if $A$ has no proper cosmall submodule. A *coclosure* of a submodule $B$ of $M$ is a cosmall submodule of $B$ in $M$ which is also a coclosed submodule of $M$. A module $M$ is called a *unique coclosure module* (denoted by UCC module) if every submodule $N$ of $M$ has a unique coclosure, $\overline{N}$, in $M$.

Following Khuri, 1979 an $R$-module $M$ is called *retractable* if $Hom_R(M,N) \neq 0$ for any nonzero submodule $N$ of $M$. This notion has been studied by some authors from various point of view. Significance of the notion has appeared in connection with study of Baer modules (Rizvi and Roman, 2004; 2007; 2009), endomorphism rings of nonsingular modules (Khuri, 1979; 1991; 2000), lattices of modules (Zhou, 1999) and semi-projective modules (Haghany and Vedadi, 2007). Dually, an $R$-module $M$ is called *coretractable* if, for any proper submodule $K$ of $M$, there exists a nonzero homomorphism $f: M \to M$ with $f(K)=0$, that is, $Hom_R(M/K, M) \neq 0$ (Clark et al., 2006).

## 2. MAIN RESULTS

The following proposition give a characterization of a coretractable module in terms of its endomorphism ring.

**Proposition 2.1.** *For a right $R$-module $M$, the following are equivalent:*

(i) $M$ is coretractable;

(ii) For $K \le N \le M$, $\ell_S(N) \le_e \ell_S(K) \Rightarrow K \xrightarrow{cs} N$ in $M$;

(iii) For any $K \langle M$, $K \xrightarrow{cs} r_M(\ell_S(K))\mathfrak{A}$ in $M$.

**Proof**.
(i)$\Rightarrow$(ii). Let $M$ be a coretractable module and $\ell_S(N) \le_e \ell_S(K)$ for $K \le N \le M$. Suppose that $N/K + L/K = M/K$, where $K \le L \le M$. If $L \ne M$, then, by hypothesis, there exists $0 \ne f \in S$ with $f(L) = 0$. Thus $f(K) = 0$ and so $0 \ne f \in \ell_S(K)$. As $\ell_S(N) \le_e \ell_S(K)$, there exists $g \in S$ such that $0 \ne gf \in \ell_S(N)$. Hence $gf(M) = gf(N + L) = 0$, which is a contradiction. Therefore $K \xrightarrow{cs} N$ in $M$.

(ii)$\Rightarrow$(iii). Let $K$ be a proper submodule of $M$. Since $\ell_S(r_M(\ell_S(K))) = \ell_S(K)$, by $(ii)$, we have $K \xrightarrow{cs} r_M(\ell_S(K))$ in $M$.

(iii)$\Rightarrow$(i) [1, Lemma 4.1].

**Theorem 2.2.** *Let $M$ be a semi-injective coretractable module. Then:*

For $K \le N \le M, K \xrightarrow{cs} N$ in $M$ if and only if $\ell_S(N) \le_e \ell_S(K)$.

**Proof**. By Proposition 2.1 we only need to prove one direction. Let $K \xrightarrow{cs} N$ in $M$. Assume that $Sf \cap \ell_S(N) = 0$ for $f \in \ell_S(K)$. Since $M$ is semi-injective, $0 = Sf \cap \ell_S(N) = \ell_S(\ker f) \cap \ell_S(N) = \ell_S(\ker f + N)$. As $M$ is coretractable, $\ker f + N = M$. Since $f \in \ell_S(K)$, $K \subseteq \ker f$. Thus $N/K + \ker f/K = M/K$. But $K \xrightarrow{CS} N$ in $M$ and so $\ker f = M$. Therefore $f = 0$. Hence $\ell_S(N) \le_e \ell_S(K)$.

Consider the following two properties:

(P1) For $K \le N \le M, K \xrightarrow{cs} N$ in $M$ if and only if $\ell_S(N) \le_e \ell_S(K)$.

(P2) For $I \le J \le S, I \le_e J$ if and only if $r_M(J) \xrightarrow{cs} r_M(I)$ in $M$.

If $M$ is a semi-injective coretractable module, then $M$ has the property (P1) (Theorem 2.2) and that $M$ has the property (P2) if and only if $I \le_e \ell_S(r_M(I))$ for any right ideal $I$ of $S$ (Corollary 2.4).

The following theorem gives the relationship between properties (P1) and (P2).

**Theorem 2.3.** *Let $M$ be an $R$-module with $S = End_R(M)$. Then:*

(i) Given (P1), then (P2) holds if and only if $I \le_e \ell_S(r_M(I))$ for each $I \le S_S$.

(ii) Given (P2), then (P1) holds if and only if $N \xrightarrow{cs} r_M(\ell_S(N))$ in $M$ for each $N \le M$.

**Proof**.

(i) Let (P1) be given. Assume that (P2) holds and let $I \le S_S$. Since $r_M(\ell_S(r_M(I)) = r_M(I)$, $r_M(\ell_S(r_M(I)) \xrightarrow{cs} r_M(I)$ in $M$. By (P2), $I \le_e \ell_S(r_M(I))$. Conversely, let $I \le_e \ell_S(r_M(I))$ for each $I \le S_S$. To prove (P2), suppose first that $I \le_e J$. Then $I \le_e \ell_S(r_M(I)) \subseteq \ell_S(r_M(J))$ and $I \le_e J \le_e \ell_S(r_M(J))$.

Therefore $I \le_e \ell_S(r_M(J))$. Hence $\ell_S(r_M(I)) \le_e \ell_S(r_M(J))$. By (P1), $r_M(J) \xrightarrow{cs} r_M(I)$ in $M$. For the other direction of (P2), let $r_M(J) \xrightarrow{cs} r_M(I)$ in $M$, where $I \le J \le S$. Using (P1), $I \le_e \ell_S(r_M(I)) \le_e \ell_S(r_M(J))$. Hence $I \le_e \ell_S(r_M(J))$. As $I \subseteq J \subseteq \ell_S(r_M(J))$, $I \le_e J$.

(ii) Let (P2) be given. If (P1) holds, then by Proposition 2.1, $N \xrightarrow{cs} r_M(\ell_S(N))$ in $M$ for every submodule $N$ of $M$. Conversely, suppose that $N \xrightarrow{cs} r_M(\ell_S(N))$ in $M$ for every $N \le M$. To prove (P1), assume first that $K \xrightarrow{cs} N$ in $M$.

Then $\ell_S(N) \subseteq \ell_S(K), K \xrightarrow{cs} r_M(\ell_S(K)) \subseteq r_M(\ell_S(N))$ and $K \xrightarrow{cs} N \xrightarrow{cs} r_M(\ell_S(N))$ in $M$. Therefore $K \xrightarrow{cs} r_M(\ell_S(N))$ and so $r_M(\ell_S(K)) \xrightarrow{cs} r_M(\ell_S(N))$ in $M$.

By (P2), $\ell_S(N) \leq_e \ell_S(K)$. For the other direction of (P1), suppose that $\ell_S(N) \leq_e \ell_S(K)$, with $K \leq N \leq M$. By using (P2), $K \xrightarrow{cs} r_M(\ell_S(K)) \xrightarrow{cs} r_M(\ell_S(N))$ in $M$, hence $K \xrightarrow{cs} r_M(\ell_S(N))$ in $M$. But $K \subseteq N \xrightarrow{cs} r_M(\ell_S(N))$, thus $K \xrightarrow{cs} N$ in $M$.

**Corollary 2.4.** *Let $M$ be a semi-injective coretractable module. Then (P2) holds if and only if $I \leq_e \ell_S(r_M(I))$ for each $I \leq S_S$.*

Set $\mathsf{C}(M) = \{N \leq M \mid N$ is a coclosed submodule of $M\}$ and $\mathsf{C}(S) = \{I \leq S_S \mid I$ is a closed right ideal of $S\}$.

**Theorem 2.5.** *Let $M$ be a semi-injective coretractable UCC module. Then the maps $N \to \ell_S(N)$ and $I \to \overline{r_M(I)}$ determine an order-preserving bijection between $\mathsf{C}(M)$ and $\mathsf{C}(S)$ if and only if $J \leq_e \ell_S(r_M(J))$ for every $J \leq S_S$.*

**Proof**. Let $J \leq_e \ell_S(r_M(J))$ for every $J \leq S_S$. By Corollary 2.4, property (P2) holds and by Theorem 2.2, property (P1) holds. Let $N \in \mathsf{C}(M)$ and $\ell_S(N) \leq_e J$. By Zorn's Lemma, we may assume that $J \in \mathsf{C}(S)$. By property (P2), $r_M(J) \xrightarrow{cs} r_M(\ell_S(N))$ in $M$. As $M$ is coretractable, $N \xrightarrow{cs} r_M(\ell_S(N))$ in $M$ (Proposition 2.1). Therefore $N = \overline{r_M(\ell_S(N))} = \overline{r_M(J)}$. Thus $N \subseteq r_M(J)$ and so $J \subseteq \ell_S(N)$. Then $\ell_S(N) = J$, that is $\ell_S$ maps $N \in \mathsf{C}(M)$ to $\ell_S(N) \in \mathsf{C}(S)$.

Let $I \in \mathcal{C}(S)$. By (P1), $\overline{r_M(I)} \xrightarrow{cs} r_M(I)$ in $M$ implies that $\ell_S(r_M(I)) \leq_e \ell_S(\overline{r_M(I)})$. Since $I \leq_e \ell_S(r_M(I)) \leq_e \ell_S(\overline{r_M(I)})$ and $I \in \mathcal{C}(S)$, $I = \ell_S(r_M(I)) = \ell_S(\overline{r_M(I)})$.

We have: $N \in \mathcal{C}(M) \to \ell_S(N) \in \mathcal{C}(S) \to \overline{r_M(\ell_S(N))} = N$, and $I \in \mathcal{C}(S) \to \overline{r_M(I)} \in \mathcal{C}(M) \to \ell_S(\overline{r_M(I)}) = I$. Therefore the two order-preserving maps are inverses of each other and so determine an order-preserving bijection between $C(M)$ and $C(S)$.

Conversely, let the map $N \to \ell_S(N)$ and $I \to \overline{r_M(I)}$ determine an order-preserving bijection between $\mathcal{C}(M)$ and $\mathcal{C}(S)$. If $I \in \mathcal{C}(S)$, then $I = \ell_S(\overline{r_M(I)})$. Since $I \leq \ell_S(r_M(I)) \leq \ell_S(\overline{r_M(I)})$, $I = \ell_S(r_M(I))$. Let $J$ be any right ideal in $S$. There is $H \in \mathcal{C}(S)$ such that $J \leq_e H$. We have $H = \ell_S(r_M(H)), r_M(H) \subseteq r_M(J)$ and $J \subseteq \ell_S(r_M(J)) \subseteq \ell_S(r_M(H)) = H$. Then $J \leq_e H$ implies that $J \leq_e \ell_S(r_M(J))$.

**Corollary 2.6.** *Let $M$ be a semi-injective coretractable UCC module and $J \leq_e \ell_S(r_M(J))$ for every $J \leq S_S$. Then $M$ is a lifting module if and only if $S_S$ is extending.*

## 3. CONCLUSION

The purpose of the current study was to find conditions under which the endomorphism ring of a semi-injective coretractable lifting module to be extending.

## REFERENCES


Amini, B., Ershad, M. and Sharif, H. (2009). Coretractable modules. *J. Aust. Math. Soc.***86**: 3: 289-304.

Clark, J., Lomp, C., Vanaja, N. and Wisbauer, R.(2006). *Lifting modules - supplements and projectivity in module theory,* Frontiers in Mathematics, Birkhäuser.

Dung, N. V., Huynh, D. V., Smith, P. F. and Wisbauer, R. (1994). *Extending Modules*, Pitman Research Notes in Mathematics Series, Longman, Harlow.

Haghany, A. and Vedadi, M. R. (2007). Study of semi-projective retractable modules.*Algebra Colloq*. **14**: 489-496.

Khuri, S. M. (1979).Endomorphism rings and lattice isomorphisms. *J. Algebra*.**59**: 2: 401-408.

Khuri, S. M. (1991). Nonsingular retractable modules and their endomorphism rings. *Bull. Austral.Math. Soc.* **43**(1): 63-71.

Khuri, S. M. (2000). The endomorphism ring of a nonsingular retractable module.*East-West J. Math.***2**: 161-170.

Rizvi, S. T. and Roman, C. S. (2004). Baer and quasi-Baer modules.*Comm. Algebra*. **32**: 103-123.

Rizvi, S. T. and. Roman, C. S. (2007).On nonsingular modules and applications.*Comm. Algebra*.**35**: 2960-2982.

Rizvi, S. T. and Roman, C. S. (2009). On direct sums of Baer modules.*J. Algebra*.**321**: 682-696.

Wisbauer, R. (1991). *Foundations of Module and Ring Theory*.Gordon and Breach, Reading.

Zhou, Z. P. (1999). A lattice isomorphism theorem for nonsingular retractable modules.*Canad.Math. Bull*.**37**(1): 140-144.